

Torsion and Supersymmetry in Ω -background

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Abstract

We study the dimensional reduction of ten-dimensional super Yang-Mills theory in curved backgrounds with torsion. We examine the parallel spinor conditions and the constraints for the torsion parameters which preserve supersymmetry and gauge symmetry in four dimensions. In particular we examine the ten-dimensional Ω -background with the torsion which is identified with the R-symmetry Wilson line gauge fields. After the dimensional reduction, we obtain the Ω -deformed $\mathcal{N} = 4$ super Yang-Mills theory. Solving the parallel spinor conditions and the torsion constraints, we classify the deformed supersymmetry associated with the topological twist of $\mathcal{N} = 4$ supersymmetry. We also study deformed supersymmetries in the Nekrasov-Shatashvili limit.

1 Introduction

The Ω -background [1] has been recognized as an interesting and useful deformation for studying non-perturbative effects in supersymmetric gauge theories via the localization technique [2, 3, 4]. This background can be embedded into superstrings and the instanton partition functions are extracted from the scattering amplitudes [5, 6, 7, 8, 9]. The microscopic deformed instanton effective action is also obtained from the D3/D(−1) brane system in the R-R 3-form backgrounds [10, 11].

The Ω -background is a curved geometry with the action of $U(1)$ vector fields and is realized in higher dimensions. The background breaks the Poincaré symmetry and also supersymmetry in general. A part of the supersymmetries, however, can be recovered by introducing the R-symmetry Wilson line gauge fields. For example, the Ω deformation of $\mathcal{N} = 2$ super Yang-Mills theory is obtained by the dimensional reduction of six-dimensional $\mathcal{N} = 1$ theory in the geometry with $U(1)^2$ -action and the $SU(2)$ R-symmetry Wilson line gauge fields. One can recover a scalar supersymmetry by choosing the appropriate Wilson lines, which is obtained by the topological twist of $\mathcal{N} = 2$ supersymmetry. Using this equivariant scalar supercharge, we can apply the localization method to compute the instanton partition function [2].

In the previous paper [12], we have studied $\mathcal{N} = 1$ super Yang-Mills theory in ten-dimensional Ω -background with the $U(1)^6$ -action and the constant $SU(4)$ R-symmetry Wilson line gauge fields. After the dimensional reduction to four dimensions, we have obtained the Ω -deformed $\mathcal{N} = 4$ super Yang-Mills theory. This theory admits the deformed supersymmetry in some cases. For the self-dual Ω -background, the theory is invariant under the anti-chiral half of the $\mathcal{N} = 4$ supersymmetry. For the ten-dimensional Ω -background restricted to the six-dimensional Ω -background with the appropriate Wilson line gauge fields, it corresponds to the $\mathcal{N} = 2^*$ deformation of $\mathcal{N} = 4$ theory. The explicit construction of the deformed supersymmetry transformations are however a very cumbersome task due to the complicated form of the deformed Lagrangian.

The purpose of the present work is to study systematically the supersymmetry of four-dimensional Ω -deformed $\mathcal{N} = 4$ super Yang-Mills theory from the viewpoint of ten-dimensional $\mathcal{N} = 1$ theory in a curved background. For the flat spacetime background

the dimensional reduction of $\mathcal{N} = 1$ supersymmetry leads to the $\mathcal{N} = 4$ supersymmetry in four dimensions [13]. The supersymmetry for the $\mathcal{N} = 1$ super Yang-Mills theory in a curved manifold leads to the parallel spinor conditions, which implies that the curved background has the Ricci-flat special holonomy. This was generalized into the curved spacetime with the Killing spinor conditions [14]. Recently the localizations of $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ theories on the sphere have been studied with the help of the supersymmetry associated with the Killing spinor conditions [15, 16].

For supersymmetry in the Ω -background, it is necessary to introduce the R-symmetry Wilson line gauge fields, which is not realized by the deformation of the metric. In order to study this Wilson line deformation, we will investigate more general set-up, namely, $\mathcal{N} = 1$ super Yang-Mills theory in a curved background with torsion. The parallel spinor conditions are modified due to the torsion, which relaxes the Ricci-flat conditions for the curved spacetime. Note that generic torsion is inconsistent with gauge invariance. We will consider a special class of torsion such that the resulting four-dimensional gauge theory is gauge invariant after the dimensional reduction. When we apply this formulation to the ten-dimensional Ω -background, we can identify the torsion with the R-symmetry Wilson line gauge fields. By solving the modified parallel spinor conditions we will find the constraints for the Ω -background parameters and the Wilson line gauge fields.

As in the $\mathcal{N} = 2$ case, the deformed scalar supersymmetries can be constructed by the topological twist of the $\mathcal{N} = 4$ supersymmetry, which is classified as follows: the half-twist, the Vafa-Witten twist and the Marcus twist [17]. We will construct the deformed supersymmetries for these twists. We will further study the Nekrasov-Shatashvili limit [18] of the deformed supersymmetry, where the supersymmetry is enhanced due to the partial recovery of the Poincaré invariance in two dimensions.

This paper is organized as follows: in section 2, we introduce the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in a curved background with torsion. In section 3, we study the dimensional reduction to four dimensions and examine the conditions such that the reduced theory has the gauge symmetry and also supersymmetry, which becomes the parallel spinor conditions and the constraints for the torsion. In section 4, we study the Ω -background with torsion, which is identified with the Wilson line gauge fields. We examine the parallel spinor conditions and the torsion constraints and obtain the conditions for

the deformation parameters. We then construct the deformed supersymmetries associated with the various twists as well as the Nekrasov-Shatashvili limit. In the appendix, we summarize the Dirac matrices in four and six dimensions.

2 Ten-dimensional super Yang-Mills theory in curved background with torsion

In this section, we introduce ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in curved background with torsion and discuss supersymmetry in the background.

We first define ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory with gauge group G in the flat spacetime. This theory contains a gauge field A_M ($M = 0, 1, \dots, 9$) and a Majorana-Weyl fermion Ψ , where both fields belong to the adjoint representation of G . The Lagrangian is

$$\mathcal{L}_0 = \frac{1}{\kappa g^2} \text{Tr} \left[-\frac{1}{4} F^{MN} F_{MN} - \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi \right], \quad (2.1)$$

where g is the coupling constant, $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$ is the field strength of A_M . The gamma matrices Γ^M are defined by $\Gamma^M \Gamma^N + \Gamma^N \Gamma^M = 2\eta^{MN}$, where the flat metric η_{MN} is taken to be Lorentzian as $\eta_{MN} = \text{diag}(-1, +1, \dots, +1)$. The gauge covariant derivative is defined by $D_M * = \partial_M * + i[A_M, *]$. We normalize the generators T^u ($u = 1, \dots, \dim G$) of the gauge group G as $\text{Tr}(T^u T^v) = \kappa \delta^{uv}$.

The Lagrangian (2.1) is invariant up to a total derivative under the supersymmetry transformation [13]

$$\delta A_M = i\bar{\zeta} \Gamma_M \Psi, \quad \delta \Psi = -\frac{1}{2} F_{MN} \Gamma^{[M} \Gamma^{N]} \zeta, \quad (2.2)$$

where ζ is a constant Majorana-Weyl spinor. The square bracket in $\Gamma^{[M} \Gamma^{N]}$ denotes the antisymmetrization of the indices in the product of two gamma matrices, defined by $\Gamma^{[M} \Gamma^{N]} = \frac{1}{2}(\Gamma^M \Gamma^N - \Gamma^N \Gamma^M)$. $\Gamma^{[M_1} \Gamma^{M_2} \dots \Gamma^{M_n]}$ is similarly normalized by the factor $1/n!$. The variation of the Lagrangian under (2.2) is

$$\begin{aligned} \delta \mathcal{L}_0 = \frac{1}{\kappa g^2} \text{Tr} \left[-\frac{1}{2} \bar{\Psi} \Gamma^M [\bar{\Psi} \Gamma_M \zeta, \Psi] + \frac{i}{2} \bar{\Psi} \Gamma^{[M} \Gamma^N \Gamma^{P]} \zeta (D_{[M} F_{NP]}) \right. \\ \left. - \frac{i}{4} D_M (\bar{\Psi} \Gamma^{[N} \Gamma^{P]} \Gamma^M \zeta F_{NP}) \right]. \end{aligned} \quad (2.3)$$

Since the first and the second terms vanish by the Fierz and the Bianchi identity, respectively and the third term is a total derivative, the action is invariant under (2.2).

We next consider ten-dimensional curved spacetime background which is represented by the metric $G_{\mathcal{MN}}$ and the torsion $T_{\mathcal{MN}}{}^{\mathcal{P}}$ (see, for example [19]). We use the calligraphic letters $\mathcal{M}, \mathcal{N}, \mathcal{P}, \dots$ ($= 0, 1, \dots, 9$) for the indices of the curved spacetime coordinates. We also introduce the vielbein $e_{\mathcal{M}}^M$, where the capital letters M, N, P, \dots are used for the indices of the tangent space coordinates. The metric is written in terms of the vielbein as $G_{\mathcal{MN}} = \eta_{MN} e_{\mathcal{M}}^M e_{\mathcal{N}}^N$. The spin connection $\hat{\omega}_{\mathcal{M},NP}$ is related to the vielbein and the torsion by Cartan's first structure equation

$$T_{\mathcal{MN}}{}^{\mathcal{P}} = \partial_{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{P}} - \partial_{\mathcal{N}} e_{\mathcal{M}}^{\mathcal{P}} + \hat{\omega}_{\mathcal{M},\mathcal{P}Q} e_{\mathcal{N}}^Q - \hat{\omega}_{\mathcal{N},\mathcal{P}Q} e_{\mathcal{M}}^Q. \quad (2.4)$$

Then $\hat{\omega}_{\mathcal{M},NP}$ is expressed in terms of $e_{\mathcal{M}}^M$ and $T_{\mathcal{MN}}{}^{\mathcal{P}}$ as

$$\hat{\omega}_{\mathcal{M},NP} = \omega_{\mathcal{M},NP} + K_{\mathcal{M},NP}, \quad (2.5)$$

$$\omega_{\mathcal{M},NP} = \frac{1}{2} (C_{\mathcal{MN},P} - C_{NP,\mathcal{M}} + C_{PM,N}), \quad (2.6)$$

$$K_{\mathcal{M},NP} = -\frac{1}{2} (T_{\mathcal{MN},P} - T_{NP,\mathcal{M}} + T_{PM,N}), \quad (2.7)$$

where $C_{\mathcal{MN}}{}^{\mathcal{P}}$ are the Ricci rotation coefficients defined by

$$C_{\mathcal{MN}}{}^{\mathcal{P}} = \partial_{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{P}} - \partial_{\mathcal{N}} e_{\mathcal{M}}^{\mathcal{P}}. \quad (2.8)$$

The tensor $K_{\mathcal{M},NP}$ is called the contorsion. The torsion is expressed in terms of the contorsion as

$$T_{\mathcal{MN}}{}^{\mathcal{P}} = -K_{\mathcal{M},N}{}^{\mathcal{P}} + K_{\mathcal{N},\mathcal{M}}{}^{\mathcal{P}}. \quad (2.9)$$

We also introduce the affine connection $\hat{\Gamma}_{\mathcal{MN}}{}^{\mathcal{P}}$. Then the covariant derivative of the vielbein is

$$\hat{\nabla}_{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{P}} = \partial_{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{P}} - \hat{\Gamma}_{\mathcal{MN}}{}^{\mathcal{P}} e_{\mathcal{P}}^{\mathcal{P}} + \hat{\omega}_{\mathcal{M},\mathcal{P}Q} e_{\mathcal{N}}^Q. \quad (2.10)$$

Here $\hat{\nabla}_{\mathcal{M}}$ denotes the spacetime covariant derivative including the torsion. We also denote the spacetime covariant derivative without the torsion as $\nabla_{\mathcal{M}}$. We relate the two connections by imposing the vielbein postulate

$$\hat{\nabla}_{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{P}} = 0. \quad (2.11)$$

From (2.5) and (2.11), $\hat{\Gamma}_{\mathcal{MN}}^{\mathcal{P}}$ is decomposed into the torsion-independent part and the contorsion part as

$$\hat{\Gamma}_{\mathcal{MN}}^{\mathcal{P}} = \Gamma_{\mathcal{MN}}^{\mathcal{P}} + K_{\mathcal{M},\mathcal{N}}^{\mathcal{P}}, \quad (2.12)$$

where the first term is the usual Christoffel symbol (the Levi-Civita connection)

$$\Gamma_{\mathcal{MN}}^{\mathcal{P}} = \frac{1}{2} G^{\mathcal{P}\mathcal{Q}} (\partial_{\mathcal{M}} G_{\mathcal{N}\mathcal{Q}} + \partial_{\mathcal{N}} G_{\mathcal{M}\mathcal{Q}} - \partial_{\mathcal{Q}} G_{\mathcal{MN}}). \quad (2.13)$$

Now we introduce the Lagrangian in the curved spacetime background with the torsion by replacing all the derivative $\partial_{\mathcal{M}}$ in (2.1) to the spacetime covariant derivative $\hat{\nabla}_{\mathcal{M}}$ and the appropriate contraction of the indices. For the vector field and the spinor field, $\hat{\nabla}_{\mathcal{M}}$ acts as

$$\hat{\nabla}_{\mathcal{M}} A_{\mathcal{N}} = \partial_{\mathcal{M}} A_{\mathcal{N}} - \hat{\Gamma}_{\mathcal{MN}}^{\mathcal{P}} A_{\mathcal{P}}, \quad \hat{\nabla}_{\mathcal{M}} \Psi = \left(\partial_{\mathcal{M}} + \frac{1}{2} \hat{\omega}_{\mathcal{M},NP} \Gamma^{NP} \right) \Psi, \quad (2.14)$$

where $\Gamma^{MN} = \frac{1}{2} \Gamma^{[M} \Gamma^{N]}$ is the ten-dimensional Lorentz generator. The field strength $F_{\mathcal{MN}}$ is replaced with $\hat{F}_{\mathcal{MN}}$ defined by

$$\begin{aligned} \hat{F}_{\mathcal{MN}} &= \hat{\nabla}_{\mathcal{M}} A_{\mathcal{N}} - \hat{\nabla}_{\mathcal{N}} A_{\mathcal{M}} + i[A_{\mathcal{M}}, A_{\mathcal{N}}] \\ &= F_{\mathcal{MN}} - T_{\mathcal{MN}}^{\mathcal{P}} A_{\mathcal{P}}. \end{aligned} \quad (2.15)$$

The gauge covariant derivative $D_{\mathcal{M}}$ is replaced with $\hat{\nabla}_{\mathcal{M}}^{(G)} * = \hat{\nabla}_{\mathcal{M}} * + i[A_{\mathcal{M}}, *]$ which is covariant with respect to both the gauge and the general coordinate transformation. Then the Lagrangian in the curved background with the torsion becomes

$$\hat{\mathcal{L}} = \frac{1}{\kappa g^2} \text{Tr} \left[-\frac{1}{4} e (e_{\mathcal{M}}^{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{N}} \hat{F}_{\mathcal{MN}})^2 - \frac{i}{2} e \bar{\Psi} e_{\mathcal{M}}^{\mathcal{M}} \Gamma^{\mathcal{M}} \hat{\nabla}_{\mathcal{M}}^{(G)} \Psi \right], \quad (2.16)$$

where e is the determinant of the vielbein and $e_{\mathcal{M}}^{\mathcal{M}}$ is the inverse vielbein. We note that (2.16) is not gauge invariant when the torsion is nonzero, since the last term in (2.15) explicitly depends on the gauge field itself.

We discuss the invariance of the action under the supersymmetry transformation:

$$\delta A_{\mathcal{M}} = i e_{\mathcal{M}}^{\mathcal{M}} \bar{\zeta} \Gamma_{\mathcal{M}} \Psi, \quad \delta \Psi = -\frac{1}{2} e_{\mathcal{M}}^{\mathcal{M}} e_{\mathcal{N}}^{\mathcal{N}} \hat{F}_{\mathcal{MN}} \Gamma^{[M} \Gamma^{N]} \zeta. \quad (2.17)$$

The variation of the Lagrangian (2.16) becomes

$$\delta\widehat{\mathcal{L}} = \frac{1}{\kappa g^2} \text{Tr} \left[\frac{i}{2} e \bar{\Psi} \Gamma^{[\mathcal{M}} \Gamma^{\mathcal{N}} \Gamma^{\mathcal{P}]} \zeta (\widehat{\nabla}_{[\mathcal{M}}^{(G)} \widehat{F}_{\mathcal{NP}]}) - \frac{i}{4} e \widehat{\nabla}_{\mathcal{M}} (\bar{\Psi} \Gamma^{[\mathcal{N}} \Gamma^{\mathcal{P}]} \Gamma^{\mathcal{M}} \zeta \widehat{F}_{\mathcal{NP}}) \right. \\ \left. + \frac{i}{2} e \bar{\Psi} \Gamma^{\mathcal{M}} \Gamma^{[\mathcal{N}} \Gamma^{\mathcal{P}]} \widehat{F}_{\mathcal{NP}} (\widehat{\nabla}_{\mathcal{M}} \zeta) \right], \quad (2.18)$$

where we have used the Fierz identity. In the first term we will compute $\widehat{\nabla}_{[\mathcal{M}}^{(G)} \widehat{F}_{\mathcal{NP}]}$. From the Bianchi identity, we obtain

$$\widehat{\nabla}_{[\mathcal{M}}^{(G)} \widehat{F}_{\mathcal{NP}]} = -(\partial_{\mathcal{Q}} A_{[\mathcal{M}}) T_{\mathcal{NP}]}{}^{\mathcal{Q}} - (\partial_{[\mathcal{M}} T_{\mathcal{NP}]}{}^{\mathcal{R}} + T_{[\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}} T_{\mathcal{P}]\mathcal{Q}}{}^{\mathcal{R}}) A_{\mathcal{R}}. \quad (2.19)$$

In order that the first term of (2.19) vanishes, the torsion must be zero. In this case the second term in (2.18) becomes a total derivative. The last term in (2.18) becomes zero by requiring that ζ satisfies the parallel spinor condition

$$\nabla_{\mathcal{M}} \zeta = 0. \quad (2.20)$$

Hence the Lagrangian (2.16) is neither invariant under the supersymmetry transformation (2.17) nor gauge invariant in ten dimensions, unless the torsion vanishes. It is related to the fact that bosonic and fermionic physical degrees of freedom are different since the gauge field becomes massive. This implies that the action is not supersymmetric. However, if we consider the dimensional reduction, the action becomes invariant under the gauge and supersymmetry transformations when the torsion satisfies certain conditions, as we will see in the next section.

3 Dimensional reduction and parallel spinor conditions

We now consider the dimensional reduction of the theory (2.16) to four dimensions. We also perform the Wick rotation $x^0 = -ix^{10}$. The local Lorentz group $SO(10)$ is reduced to $SU(2)_L \times SU(2)_R \times SU(4)$, where $SU(2)_L \times SU(2)_R$ is the Lorentz group in four dimensions and $SU(4)$ becomes the R-symmetry of the reduced theory. After the dimensional reduction, the gauge field $A_{\mathcal{M}}$ is decomposed as $A_{\mathcal{M}} = (A_{\mu}, \varphi_{\mathcal{A}})$, where A_{μ} ($\mu = 1, \dots, 4$) is the gauge field and $\varphi_{\mathcal{A}}$ ($\mathcal{A} = 5, \dots, 10$) are the scalar fields in four dimensions. The

spinor field Ψ is also decomposed as $\Psi = (\Lambda_\alpha^A, \bar{\Lambda}_A^{\dot{\alpha}})$, where $\alpha, \dot{\alpha} = 1, 2$ are the $SU(2)_L$ and $SU(2)_R$ indices respectively. These indices are raised and lowered by the antisymmetric ε -symbol normalized as $\varepsilon^{12} = -\varepsilon_{12} = 1$. $A = 1, \dots, 4$ is the index for the vector representation of $SU(4)$. The gamma matrices are decomposed as

$$\Gamma^M = \left(-i \begin{pmatrix} 0 & \sigma_{\alpha\dot{\alpha}}^m \\ \bar{\sigma}^{m\dot{\alpha}\alpha} & 0 \end{pmatrix} \otimes \mathbf{1}_8, \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Sigma^{aAB} \\ \bar{\Sigma}_{AB}^a & 0 \end{pmatrix} \right), \quad (3.1)$$

where we have decomposed the index M as $M = (m, a)$ ($m = 1, \dots, 4$, $a = 5, \dots, 10$). $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. The conventions of four- and six-dimensional Dirac matrices $\sigma^m, \bar{\sigma}^m, \Sigma^a, \bar{\Sigma}^a$ are given in the appendix.

The fields and the background do not depend on the internal coordinates x^A in the dimensional reduction. By setting $\partial_A = 0$ in (2.16), we obtain the four-dimensional Lagrangian as

$$\begin{aligned} \hat{\mathcal{L}}_{4D} = & \frac{1}{\kappa g^2} \text{Tr} \left[\frac{1}{4} e \left(e_m^\mu e_n^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_n^A - e_n^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_n^B \hat{F}_{AB} \right)^2 \right. \\ & + \frac{1}{2} e \left(e_m^\mu e_a^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_a^A - e_a^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_a^B \hat{F}_{AB} \right)^2 \\ & + \frac{1}{4} e \left(e_a^\mu e_b^\nu \hat{F}_{\mu\nu} + (e_a^\mu e_b^A - e_b^\mu e_a^A) \hat{F}_{\mu A} + e_a^A e_b^B \hat{F}_{AB} \right)^2 \\ & + e_m^\mu \Lambda^{\alpha A} \sigma_{\alpha\dot{\alpha}}^m D_\mu \bar{\Lambda}_A^{\dot{\alpha}} + i e_m^A \Lambda^{\alpha A} \sigma_{\alpha\dot{\alpha}}^m [\varphi_A, \bar{\Lambda}_A^{\dot{\alpha}}] \\ & - \frac{1}{2} e_a^A \bar{\Sigma}_{AB}^a \Lambda^{\alpha A} [\varphi_A, \Lambda_B^{\dot{\alpha}}] - \frac{1}{2} e_a^A \Sigma^{aAB} \bar{\Lambda}_{\dot{\alpha}A} [\varphi_A, \Lambda_B^{\dot{\alpha}}] \\ & + \frac{i}{2} e_a^\mu \bar{\Sigma}_{AB}^a \Lambda^{\alpha A} D_\mu \Lambda_\alpha^B + \frac{i}{2} e_a^\mu \Sigma^{aAB} \bar{\Lambda}_{\dot{\alpha}A} D_\mu \bar{\Lambda}_B^{\dot{\alpha}} + \frac{1}{4} \hat{\omega}_{m,np} \Lambda^A \epsilon^{mnpq} \sigma_q \bar{\Lambda}_A \\ & + \frac{i}{4} (\hat{\omega}_{a,mn} + 2\hat{\omega}_{m,na}) (\Lambda^A \sigma^{mn} \bar{\Sigma}_{AB}^a \Lambda^B + \bar{\Lambda}_A \bar{\sigma}^{mn} \Sigma^{aAB} \bar{\Lambda}_B) \\ & + \frac{1}{2} (2\hat{\omega}_{a,mb} - \hat{\omega}_{m,ab}) \Lambda^A \sigma^m (\bar{\Sigma}^{ab})_A{}^B \bar{\Lambda}_B \\ & \left. - \frac{i}{8} \hat{\omega}_{a,bc} (\Lambda^A (\bar{\Sigma}^{[a} \Sigma^b \bar{\Sigma}^{c]})_{AB} \Lambda^B + \bar{\Lambda}_A (\Sigma^{[a} \bar{\Sigma}^b \Sigma^{c]})^{AB} \Lambda_B) \right], \quad (3.2) \end{aligned}$$

where ϵ^{mnpq} is the totally antisymmetric tensor normalized as $\epsilon^{1234} = 1$. σ^{mn} , $\bar{\sigma}^{mn}$ and Σ^{ab} , $\bar{\Sigma}^{ab}$ are the Lorentz generators in four and six dimensions respectively, which are defined in the appendix. $\hat{F}_{\mu\nu}$, $\hat{F}_{\mu A}$, \hat{F}_{AB} are the components of the modified field strength

(2.15), which are obtained as

$$\begin{aligned}\widehat{F}_{\mu\nu} &= F_{\mu\nu} - T_{\mu\nu}{}^\rho A_\rho - T_{\mu\nu}{}^{\mathcal{A}} \varphi_{\mathcal{A}}, \\ \widehat{F}_{\mu\mathcal{A}} &= D_\mu \varphi_{\mathcal{A}} - T_{\mu\mathcal{A}}{}^\rho A_\rho - T_{\mu\mathcal{A}}{}^{\mathcal{B}} \varphi_{\mathcal{B}}, \\ \widehat{F}_{\mathcal{A}\mathcal{B}} &= i[\varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}] - T_{\mathcal{A}\mathcal{B}}{}^\rho A_\rho - T_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} \varphi_{\mathcal{C}}.\end{aligned}\tag{3.3}$$

The Lagrangian (3.2) does not have the gauge invariance since $\widehat{\mathcal{L}}_{4\text{D}}$ depends on A_μ explicitly. However the gauge invariance is recovered by setting

$$T_{\mathcal{MN}}{}^\rho = (T_{\mu\nu}{}^\rho, T_{\mu\mathcal{A}}{}^\rho, T_{\mathcal{AB}}{}^\rho) = 0.\tag{3.4}$$

Next we examine the supersymmetry in the dimensionally reduced theory under the gauge invariance condition (3.4). We use the notation in ten dimensions for convenience. The condition for supersymmetry is that (2.18) becomes a total derivative. (2.19) vanishes due to (3.4) and the reduction $\partial_{\mathcal{A}} = 0$. The second and the third terms become zero by imposing the condition

$$\partial_{[\mathcal{M}} T_{\mathcal{NP}]}{}^{\mathcal{R}} + T_{[\mathcal{MN}}{}^{\mathcal{Q}} T_{\mathcal{P}]\mathcal{Q}}{}^{\mathcal{R}} = 0.\tag{3.5}$$

The second term in the variation (2.18) does not become a total derivative. We have

$$e \widehat{\nabla}_{\mathcal{M}} V^{\mathcal{M}} = \partial_{\mathcal{M}}(e V^{\mathcal{M}}) + e T_{\mathcal{MN}}{}^{\mathcal{M}} V^{\mathcal{N}},\tag{3.6}$$

where the vector $V^{\mathcal{M}}$ is given by

$$V^{\mathcal{M}} = \frac{1}{\kappa g^2} \text{Tr} \left[-\frac{i}{4} \bar{\Psi} \Gamma^{\mathcal{NP}} \Gamma^{\mathcal{M}} \zeta \widehat{F}_{\mathcal{NP}} \right].\tag{3.7}$$

Hence we have the traceless condition such that the second term in (3.6) vanishes as

$$T_{\mathcal{MN}}{}^{\mathcal{M}} = 0.\tag{3.8}$$

The last term in (2.18) becomes zero when ζ satisfies the parallel spinor condition modified by the torsion as

$$\widehat{\nabla}_{\mathcal{M}} \zeta = 0.\tag{3.9}$$

Therefore the dimensionally reduced theory from (2.16) is invariant under the supersymmetry transformation (2.17) generated by the parallel spinor ζ satisfying (3.9) when

the torsion $T_{\mathcal{MN}}{}^{\mathcal{P}}$ satisfies the conditions (3.4), (3.5) and (3.8). After the dimensional reduction, the supersymmetry transformation (2.17) becomes

$$\begin{aligned}
\delta A_\mu &= -e_\mu^m \zeta^A \sigma_m \bar{\Lambda}_A - e_\mu^m \bar{\zeta}^A \bar{\sigma}_m \Lambda_A + i e_\mu^A \zeta^A \bar{\Sigma}_{aAB} \Lambda^B - i e_\mu^A \bar{\zeta}_A \Sigma_a^{AB} \bar{\Lambda}_B, \\
\delta \Lambda^A &= \sigma^{mn} \zeta^A \left(e_m^\mu e_n^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_n^A - e_n^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_n^B \hat{F}_{AB} \right) \\
&\quad + i \Sigma^{aAB} \sigma^m \bar{\zeta}_B \left(e_m^\mu e_a^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_a^A - e_a^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_a^B \hat{F}_{AB} \right) \\
&\quad + (\Sigma^{ab})^A{}_B \zeta^B \left(e_a^\mu e_b^\nu \hat{F}_{\mu\nu} + (e_a^\mu e_b^A - e_b^\mu e_a^A) \hat{F}_{\mu A} + e_a^A e_b^B \hat{F}_{AB} \right), \\
\delta \bar{\Lambda}_A &= \bar{\sigma}^{mn} \bar{\zeta}_A \left(e_m^\mu e_n^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_n^A - e_n^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_n^B \hat{F}_{AB} \right) \\
&\quad + i \bar{\Sigma}_{AB}^a \bar{\sigma}^m \zeta^B \left(e_m^\mu e_a^\nu \hat{F}_{\mu\nu} + (e_m^\mu e_a^A - e_a^\mu e_m^A) \hat{F}_{\mu A} + e_m^A e_a^B \hat{F}_{AB} \right) \\
&\quad + (\bar{\Sigma}^{ab})_A{}^B \bar{\zeta}_B \left(e_a^\mu e_b^\nu \hat{F}_{\mu\nu} + (e_a^\mu e_b^A - e_b^\mu e_a^A) \hat{F}_{\mu A} + e_a^A e_b^B \hat{F}_{AB} \right), \\
\delta \varphi_A &= -e_A^m \zeta^A \sigma_m \bar{\Lambda}_A - e_A^m \bar{\zeta}^A \bar{\sigma}_m \Lambda_A + i e_A^a \zeta^A \bar{\Sigma}_{aAB} \Lambda^B - i e_A^a \bar{\zeta}_A \Sigma_a^{AB} \bar{\Lambda}_B, \tag{3.10}
\end{aligned}$$

where we decomposed ζ as $\zeta = (\zeta_\alpha^A, \bar{\zeta}_{\dot{\alpha}A})$.

Here we consider the case of the flat spacetime and that the constant torsion is turned only in the six-dimensional direction as an example. In the Lagrangian (3.2), the torsion gives the mass terms for the fermions and scalars. The supersymmetry conditions restrict the form of the mass terms. From (3.5) the constant torsion satisfies $T_{[ab}{}^d T_{c]d}{}^e = 0$, which implies that the torsion can be regarded as the structure constant of a Lie algebra \mathcal{T} and forms the adjoint representation of \mathcal{T} . The dimension of \mathcal{T} is equal to or less than six, where the latter case is possible when the components of the torsion are not linearly independent. The traceless condition $T_{ab}{}^a = 0$ from (3.8) must be also satisfied.

If the torsion is totally antisymmetric, the traceless condition is satisfied. Moreover, from (2.7) the contorsion is proportional to the torsion as $K_{a,bc} = -\frac{1}{2} T_{ab,c}$. In this case \mathcal{T} becomes a subalgebra of $SU(4)$. The parallel spinor condition (3.9) becomes

$$T_{ab,c} (\Sigma^{bc})^A{}_B \zeta_\alpha^B = 0, \quad T_{ab,c} (\bar{\Sigma}^{bc})_A{}^B \bar{\zeta}_{\dot{\alpha}B} = 0. \tag{3.11}$$

Since the matrices acting on the parallel spinors in (3.11) are the Hermitian conjugate to each other, we have the same number of the left-handed and the right-handed parallel spinors. The number of supersymmetries depends on the choice of \mathcal{T} and how \mathcal{T} is embedded into $SU(4)$. We summarize the relation between \mathcal{T} and the number of supersymmetry in table 1.

\mathcal{T}	# of SUSY
$SU(2) \times SU(2), SU(2) \times U(1)^2, U(1)^3$	$\mathcal{N} = 0$
$SU(2) \times U(1), U(1)^2$	$\mathcal{N} = 1$
$SU(2), U(1)$	$\mathcal{N} = 2$

Table 1: Torsion algebra $\mathcal{T} \in SU(4)$ and the number \mathcal{N} of supersymmetry. The embedding of \mathcal{T} is chosen such that \mathcal{N} becomes maximal.

As an example, we consider the case $\mathcal{T} = U(1)^2$. If \mathcal{T} is embedded into $SU(4)$ appropriately, the parallel spinor condition (3.11) has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{1a} & 0 & 0 \\ 0 & 0 & t_{2a} & 0 \\ 0 & 0 & 0 & t_{3a} \end{pmatrix}^A \zeta^B = 0, \quad t_{1a} + t_{2a} + t_{3a} = 0, \quad t_{1a}, t_{2a}, t_{3a} \neq 0. \quad (3.12)$$

The similar condition for $\bar{\zeta}_A$ holds. The solution to (3.12) is $\zeta^A = (\zeta^1, 0, 0, 0)^T$. Then we have $\mathcal{N} = 1$ supersymmetry. We can check that it corresponds to the $\mathcal{N} = 1^*$ deformation [20]. The mass term for the fermions takes the form of

$$\mathcal{L}_m = m_{AB} \Lambda^A \Lambda^B + \bar{m}^{AB} \bar{\Lambda}_A \bar{\Lambda}_B, \quad (3.13)$$

where the two mass matrices m_{AB} and \bar{m}^{AB} are defined by

$$m_{AB} = \frac{i}{16} (\bar{\Sigma}^{[a} \Sigma^b \bar{\Sigma}^{c]})_{AB} T_{ab,c}, \quad \bar{m}^{AB} = \frac{i}{16} (\Sigma^{[a} \bar{\Sigma}^b \Sigma^{c]})^{AB} T_{ab,c}. \quad (3.14)$$

When the parallel spinor condition becomes the form of (3.12), we can show that each mass matrix has one zero eigenvalue. This is the $\mathcal{N} = 1^*$ deformation. Similarly, in the case of $\mathcal{T} = U(1)$ we obtain the $\mathcal{N} = 2^*$ deformation.

4 Ω -background and Deformed Supersymmetry

In this section, we study the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in the Ω -background with torsion. We solve the parallel spinor and the torsion conditions obtained in the previous section and classify the supersymmetries.

4.1 $\mathcal{N} = 4$ super Yang-Mills theory in Ω -background

The Lagrangian of the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in the Ω -background is obtained by the dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in the spacetime with the metric:

$$\begin{aligned} ds^2 &= (dx^m + \Omega^m_a dx^a)^2 + (dx^a)^2, \\ \Omega^m_a &= \Omega^{mn}_a x_n, \quad \Omega^{mn}_a = -\Omega^{nm}_a, \end{aligned} \tag{4.1}$$

where x^m and x^a are the coordinates of the four- and six-dimensional spaces. The anti-symmetric matrices Ω_{mna} are parameterized as

$$\Omega_{mna} = \begin{pmatrix} 0 & \epsilon_{1a} & 0 & 0 \\ -\epsilon_{1a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{2a} \\ 0 & 0 & \epsilon_{2a} & 0 \end{pmatrix}, \tag{4.2}$$

where $\epsilon_{1a}, \epsilon_{2a}$ are real parameters. These matrices commute with each other

$$\Omega_m^p{}_a \Omega_{pnb} - \Omega_m^p{}_b \Omega_{pna} = 0. \tag{4.3}$$

The vielbein is given by

$$\begin{aligned} e^M_{\mathcal{M}} &= \begin{pmatrix} e^m_{\mu} & e^a_{\mu} \\ e^m_{\mathcal{A}} & e^a_{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} \delta^m_{\mu} & 0 \\ \delta^a_{\mathcal{A}} \Omega^m_a & \delta^a_{\mathcal{A}} \end{pmatrix}, \\ e^{\mathcal{M}}_M &= \begin{pmatrix} e^{\mu}_m & e^{\mathcal{A}}_m \\ e^{\mu}_a & e^{\mathcal{A}}_a \end{pmatrix} = \begin{pmatrix} \delta^{\mu}_m & 0 \\ -\delta^{\mu}_m \Omega^m_a & \delta^{\mathcal{A}}_a \end{pmatrix}. \end{aligned} \tag{4.4}$$

We introduce the constant torsion along the internal directions, which is consistent with the dimensional reduction. We want to study the supersymmetry conditions in this setup. Another way to recover parts of supersymmetry is to introduce the constant Wilson line gauge field \mathcal{A}_a by gauging the $SU(4)$ R-symmetry [4, 12]. From the expression of the covariant derivative (2.14) in the general curved background, we find that the $SU(4)$ R-symmetry Wilson line gauge field is identified with the contorsion through the following relation,

$$K_{\mathcal{A},bc} = -i\delta_{\mathcal{A}}^a (\mathcal{A}_a)^A{}_B (\Sigma_{bc})^B{}_A, \tag{4.5}$$

or equivalently

$$(\mathcal{A}_a)^A{}_B = \frac{i}{4} \delta_a^A (\Sigma_{bc})^A{}_B K_{\mathcal{A},bc}. \quad (4.6)$$

The other components except $K_{\mathcal{A},bc}$ are zero. From (2.9), the non-trivial components of the torsion are given by

$$T_{\mathcal{A}\mathcal{B}}{}^\mu = -\delta_{\mathcal{A}}^a \delta_{\mathcal{B}}^b (K_{a,b}{}^c - K_{b,a}{}^c) \Omega_c^\mu, \quad (4.7)$$

$$T_{\mathcal{A}\mathcal{B}}{}^c = -\delta_{\mathcal{A}}^a \delta_{\mathcal{B}}^b \delta_c^c (K_{a,b}{}^c - K_{b,a}{}^c). \quad (4.8)$$

The non-zero components of the spin and affine connections are evaluated as

$$\widehat{\omega}_{\mathcal{A},mn} = \delta_{\mathcal{A}}^a \Omega_{mna}, \quad \widehat{\omega}_{\mathcal{A},bc} = K_{\mathcal{A},bc}, \quad (4.9)$$

$$\widehat{\Gamma}_{\mu\mathcal{A}}{}^\nu = \Omega^\nu{}_{\mu\mathcal{A}}, \quad (4.10)$$

$$\widehat{\Gamma}_{\mathcal{A}\mathcal{B}}{}^\mu = \Omega^\mu{}_{\rho\mathcal{A}} \Omega^\rho{}_{\mathcal{B}} - \delta_{\mathcal{B}}^b \Omega_c^\mu K_{\mathcal{A},b}{}^c, \quad (4.11)$$

$$\widehat{\Gamma}_{\mathcal{A}\mathcal{B}}{}^c = \delta_{\mathcal{B}}^b \delta_c^c K_{\mathcal{A},b}{}^c. \quad (4.12)$$

The gauge invariance condition (3.4) reads

$$T_{\mathcal{A}\mathcal{B}}{}^\mu = 0. \quad (4.13)$$

Using this condition and substituting the vielbein (4.4) and the torsion (4.7), (4.8) into the Lagrangian (3.2), we obtain

$$\begin{aligned} \mathcal{L}_{(\Omega,\mathcal{A})} = & \frac{1}{\kappa g^2} \text{Tr} \left[\frac{1}{4} F^{mn} F_{mn} + \Lambda^A \sigma^m D_m \bar{\Lambda}_A + \frac{1}{2} (D_m \varphi_a - F_{mn} \Omega_a^n)^2 \right. \\ & - \frac{1}{2} (\Sigma_a)^{AB} \bar{\Lambda}_A [\varphi_a, \bar{\Lambda}_B] - \frac{1}{2} (\bar{\Sigma}_a)_{AB} \Lambda^A [\varphi_a, \Lambda^B] \\ & - \frac{1}{4} \left([\varphi_a, \varphi_b] + i \Omega_a^m D_m \varphi_b - i \Omega_b^m D_m \varphi_a - i F_{mn} \Omega_a^m \Omega_b^n \right. \\ & \quad \left. - \frac{1}{2} \left((\Sigma_b \bar{\Sigma}_c)^A{}_B \varphi_c (\mathcal{A}_a)^B{}_A - (\Sigma_a \bar{\Sigma}_c)^A{}_B \varphi_c (\mathcal{A}_b)^B{}_A \right) \right)^2 \\ & - \frac{i}{2} \Omega_a^m \left((\Sigma_a)^{AB} \bar{\Lambda}_A D_m \bar{\Lambda}_B + (\bar{\Sigma}_a)_{AB} \Lambda^A D_m \Lambda^B \right) \\ & + \frac{i}{4} \Omega_{mna} \left((\Sigma_a)^{AB} \bar{\Lambda}_A \bar{\sigma}^{mn} \bar{\Lambda}_B + (\bar{\Sigma}_a)_{AB} \Lambda^A \sigma^{mn} \Lambda^B \right) \\ & \left. + \frac{1}{2} (\Sigma_a)^{AB} \bar{\Lambda}_A \bar{\Lambda}_D (\mathcal{A}_a)^D{}_B - \frac{1}{2} (\bar{\Sigma}_a)_{AB} \Lambda^A (\mathcal{A}_a)^B{}_D \Lambda^D \right]. \quad (4.14) \end{aligned}$$

This Lagrangian indeed coincides with the Ω -deformed one with the R-symmetry Wilson line obtained in [12].

4.2 Supersymmetry conditions in Ω -background

Now we examine the supersymmetry and gauge invariance conditions for the Lagrangian (4.14). We first write down the parallel spinor condition (3.9) in the Ω -background with the torsion. Then, we consider the constraints on the torsion (3.4), (3.5) and (3.8).

4.2.1 Parallel spinor condition

The parallel spinor condition (3.9) in the Ω -background with the torsion is given by

$$\widehat{\nabla}_\mu \zeta = \partial_\mu \zeta = 0, \quad (4.15)$$

$$\widehat{\nabla}_A \zeta = \frac{1}{4} \delta^a_A (\Omega_{mna} \Gamma^{mn} + K_{a,bc} \Gamma^{bc}) \zeta = 0. \quad (4.16)$$

From the condition (4.15), the parameter ζ becomes constant. When the Ω -background matrices Ω_{mna} are anti-self-dual or self-dual, and the torsion is zero, the condition (4.16) is satisfied for the chiral or anti-chiral parameters $\zeta_\alpha^A, \bar{\zeta}_A^{\dot{\alpha}}$ respectively. In these cases, all the torsion conditions (3.4), (3.8) and (3.5) are trivially satisfied and half of the $\mathcal{N} = 4$ supersymmetries are preserved [12]. However, when Ω_{mna} is not (anti-)self-dual, the condition (4.16) can not be satisfied in general.

In following, we consider Ω_{mna} which is not (anti-)self-dual. Since Γ^{mn} and Γ^{ab} are generators of four- and six-dimensional rotations, Ω_{mna} and $K_{a,bc}$ are rotational parameters of $SO(4)$ and $SO(6)$. Since the matrices Ω_{mna} commute with each other, they are the rotational parameters of the $U(1)_L \times U(1)_R$ Cartan subgroup of the four-dimensional Lorentz group $SO(4) \simeq SU(2)_L \times SU(2)_R$. For the six-dimensional rotation group, we consider the subgroup $SO(2)' \times SO(4)' \simeq U(1)' \times SU(2)_{L'} \times SU(2)_{R'}$ of $SO(6)$. We decompose the six-dimensional vector index a into $a = (a', \hat{a})$ ($a' = 5, 6, \hat{a} = 7, 8, 9, 10$), associated with the $SO(2)'$ and $SO(4)'$ rotations respectively. We cancel parts of the component in $\Omega_{mna} \Gamma^{mn}$ and $K_{a,bc} \Gamma^{bc}$ by identifying $U(1)$ charges of the Lorentz group with that of six dimensions. This is done by identifying $SU(2)$'s in the Lorentz group with those in the R-symmetry group. These identifications correspond to the topological twist of the four-dimensional $\mathcal{N} = 4$ supersymmetry [17]. Then the components of the contorsion $K_{a,bc}$ are the parameters of the Cartan subgroup $U(1)_{L'} \times U(1)_{R'}$ of $SU(2)_{L'} \times SU(2)_{R'}$,

	$U(1)_L$	$U(1)_R$	$U(1)_{L'}$	$U(1)_{R'}$
$\zeta_\alpha^{A'}$	$\pm\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\zeta_\alpha^{\hat{A}}$	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0
$\bar{\zeta}_{A'}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$
$\bar{\zeta}_{\hat{A}}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0

Table 2: $U(1)_X$ charges of spinors.

where $K_{a,\hat{b}\hat{c}}$ are non-zero and

$$K_{a,b'c'} = K_{a,b'\hat{c}} = 0. \quad (4.17)$$

The condition (4.16) becomes

$$(\Omega_{mna}\Gamma^{mn} + K_{a,\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}})\zeta = 0. \quad (4.18)$$

Now we determine the $U(1)$ charges of the spinor parameter $\zeta = (\zeta_\alpha^A, \bar{\zeta}_A^{\dot{\alpha}})$. The representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$ are decomposed into the representation of $U(1)' \times SU(2)_{L'} \times SU(2)_{R'}$:

$$\mathbf{4} = (\mathbf{2}, \mathbf{1})_{1/2} + (\mathbf{1}, \mathbf{2})_{-1/2}, \quad \bar{\mathbf{4}} = (\mathbf{2}, \mathbf{1})_{-1/2} + (\mathbf{1}, \mathbf{2})_{1/2}, \quad (4.19)$$

where the first and the second components in the parenthesis are the representations of $SU(2)_{L'}$ and $SU(2)_{R'}$ respectively. The subscript $\pm 1/2$ denotes the $U(1)'$ charge and $\mathbf{2}, \mathbf{1}$ are the two-dimensional and the trivial representation of each $SU(2)$. Then we have the following decomposition of spinors,

$$\zeta_\alpha^A = (\zeta_\alpha^{A'}, \zeta_\alpha^{\hat{A}}), \quad \bar{\zeta}_A^{\dot{\alpha}} = (\bar{\zeta}_{A'}^{\dot{\alpha}}, \bar{\zeta}_{\hat{A}}^{\dot{\alpha}}), \quad A' = 1, 2, \quad \hat{A} = 3, 4. \quad (4.20)$$

Spinors that have $A' = 1, 2$ are $\mathbf{2}$ representation of $SU(2)_{R'}$ while that have $\hat{A} = 3, 4$ are $\mathbf{2}$ representation of $SU(2)_{L'}$. The generators of $SO(4)'$ for the spinors $\zeta_\alpha^{A'}, \bar{\zeta}_{A'}^{\dot{\alpha}}$ and $\zeta_{\hat{A}}^\alpha, \bar{\zeta}_{\hat{A}}^{\dot{\alpha}}$ are $(\Sigma_{ab})_{A'}^{B'} = -(\bar{\Sigma}_{ab})^{B'}_{A'}$ and $(\Sigma_{ab})_{\hat{A}}^{\hat{B}} = -(\bar{\Sigma}_{ab})^{\hat{B}}_{\hat{A}}$. We are interested in the $U(1)_X$ ($X = L, R, L', R'$) charges of spinors associated with the Cartan subgroups of $SU(2)_X$. We summarize the $U(1)$ charges for each spinor in table 2.

There are three topological twists called the half twist [17], the Vafa-Witten twist [21] and the Marcus twist [22].

Half twist In the half twist, $SU(2)_{R'}$ and $SU(2)_R$ are identified while the $SU(2)_{L'}$ and $SU(2)_L$ are left intact. The new Lorentz group is defined as $SU(2)_L \times [SU(2)_{R'} \times SU(2)_R]_{\text{diag}}$ where $[SU(2)_{R'} \times SU(2)_R]_{\text{diag}}$ denotes the diagonal subgroup of $SU(2)_{R'} \times SU(2)_R$. Spinors $\zeta_\alpha^{A'}$, $\bar{\zeta}_{A'}^{\dot{\alpha}}$ can be decomposed under the new Lorentz group as

$$\zeta_\alpha^{A'} = (\sigma^m)_{\alpha B'} \epsilon^{A' B'} \zeta_m, \quad \bar{\zeta}_{A'}^{\dot{\alpha}} = \delta_{A'}^{\dot{\alpha}} \bar{\zeta} + (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{A'} \bar{\zeta}_{mn}, \quad (4.21)$$

where ζ_m , $\bar{\zeta}$ and $\bar{\zeta}_{mn}$ are vector, scalar and anti-self-dual tensor respectively.

Vafa-Witten twist In the Vafa-Witten twist, $SU(2)_{L'} \times SU(2)_{R'}$ in the R-symmetry and $SU(2)_R$ in the Lorentz symmetry is identified. The new Lorentz group is defined as $SU(2)_L \times [SU(2)_{L'} \times SU(2)_{R'} \times SU(2)_R]_{\text{diag}}$. Spinors are decomposed as

$$\zeta_\alpha^{A'} = (\sigma^m)_{\alpha B'} \epsilon^{A' B'} \zeta_m, \quad \bar{\zeta}_{A'}^{\dot{\alpha}} = \delta_{A'}^{\dot{\alpha}} \bar{\zeta} + (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{A'} \bar{\zeta}_{mn}, \quad (4.22)$$

$$\zeta_\alpha^{\hat{A}} = (\sigma^m)_{\alpha \hat{B}} \epsilon^{\hat{A} \hat{B}} \zeta_m', \quad \bar{\zeta}_{\hat{A}}^{\dot{\alpha}} = \delta_{\hat{A}}^{\dot{\alpha}} \bar{\zeta}' + (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\hat{A}} \bar{\zeta}_{mn}'. \quad (4.23)$$

Marcus twist In the Marcus twist, $SU(2)_{L'}$ and $SU(2)_L$, $SU(2)_{R'}$ and $SU(2)_R$ are identified. The new Lorentz group is defined as $[SU(2)_{L'} \times SU(2)_L]_{\text{diag}} \times [SU(2)_{R'} \times SU(2)_R]_{\text{diag}}$. Spinors are decomposed as

$$\zeta_\alpha^{A'} = (\sigma^m)_{\alpha B'} \epsilon^{A' B'} \zeta_m, \quad \bar{\zeta}_{A'}^{\dot{\alpha}} = \delta_{A'}^{\dot{\alpha}} \bar{\zeta} + (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{A'} \bar{\zeta}_{mn}, \quad (4.24)$$

$$\zeta_\alpha^{\hat{A}} = \delta_\alpha^{\hat{A}} \zeta + (\sigma^{mn})_\alpha^{\hat{A}} \zeta_{mn}, \quad \bar{\zeta}_{\hat{A}}^{\dot{\alpha}} = (\bar{\sigma}^m)^{\dot{\alpha} \hat{B}} \epsilon_{\hat{A} \hat{B}} \bar{\zeta}_m. \quad (4.25)$$

4.2.2 Torsion conditions

We have obtained the parallel spinor condition (4.18). Now we write down the conditions on the torsion (3.4), (3.5) and (3.8) in the Ω -background. The gauge invariance condition (3.4) reads

$$(K_{a,b}{}^c - K_{b,a}{}^c) \Omega_{mnc} = 0. \quad (4.26)$$

The condition (3.5) becomes

$$T_{[ab}{}^d T_{c]d}{}^e = 0, \quad (4.27)$$

while the condition (3.8) is reduced to

$$T_{ab}{}^a = -K_{a,b}{}^a = 0. \quad (4.28)$$

We first consider the condition (4.26). From (4.17), the condition (4.26) becomes

$$K_{a',\hat{b}}{}^{\hat{c}}\Omega_{mn\hat{c}} = 0, \quad (4.29)$$

$$(K_{\hat{a},\hat{b}}{}^{\hat{c}} - K_{\hat{b},\hat{a}}{}^{\hat{c}})\Omega_{mn\hat{c}} = 0. \quad (4.30)$$

Since these conditions are independent of $\Omega_{mna'}$, we assume that the parameters $\epsilon_{1a'}, \epsilon_{2a'}$ are non-zero without loss of generality.

Next, the condition (4.27) becomes

$$T_{a'b'}{}^{\hat{d}}T_{\hat{c}\hat{d}}{}^{\hat{e}} + T_{b'\hat{c}}{}^{\hat{d}}T_{a'\hat{d}}{}^{\hat{e}} + T_{\hat{c}a'}{}^{\hat{d}}T_{b'\hat{d}}{}^{\hat{e}} = 0, \quad (4.31)$$

$$T_{a'\hat{b}}{}^{\hat{d}}T_{\hat{c}\hat{d}}{}^{\hat{e}} + T_{\hat{b}\hat{c}}{}^{\hat{d}}T_{a'\hat{d}}{}^{\hat{e}} + T_{\hat{c}a'}{}^{\hat{d}}T_{\hat{b}\hat{d}}{}^{\hat{e}} = 0, \quad (4.32)$$

$$T_{\hat{a}\hat{b}}{}^{\hat{d}}T_{\hat{c}\hat{d}}{}^{\hat{e}} + T_{\hat{b}\hat{c}}{}^{\hat{d}}T_{\hat{a}\hat{d}}{}^{\hat{e}} + T_{\hat{c}\hat{a}}{}^{\hat{d}}T_{\hat{b}\hat{d}}{}^{\hat{e}} = 0. \quad (4.33)$$

Using (2.9), the conditions (4.31) and (4.32) are rewritten as

$$K_{b',\hat{c}}{}^{\hat{d}}K_{a',\hat{d}}{}^{\hat{e}} - K_{a',\hat{c}}{}^{\hat{d}}K_{b',\hat{d}}{}^{\hat{e}} = 0, \quad (4.34)$$

$$K_{a',\hat{b}}{}^{\hat{d}}T_{\hat{c}\hat{d}}{}^{\hat{e}} + K_{a',\hat{c}}{}^{\hat{d}}T_{\hat{b}\hat{d}}{}^{\hat{e}} - K_{a',\hat{d}}{}^{\hat{e}}T_{\hat{b}\hat{c}}{}^{\hat{d}} = 0. \quad (4.35)$$

The first equation (4.34) is the commuting condition of the contorsion matrices $K_{a',\hat{b}}{}^{\hat{c}}$. Then $K_{a',\hat{b}\hat{c}}$ are parameters of $U(1)_{L'} \times U(1)_{R'}$. Finally, the condition (4.28) becomes

$$K_{\hat{a},\hat{b}}{}^{\hat{a}} = 0. \quad (4.36)$$

In the following we solve the parallel spinor condition and the conditions on the torsion in each twist separately.

4.3 Solutions to the conditions

4.3.1 Half twist

First we consider the half twist. The supercharges $Q_{\alpha}^{A'}$ and $\bar{Q}_{A'}^{\dot{\alpha}}$ are decomposed into \bar{Q} , Q_m , \bar{Q}_{mn} . The parameters $\zeta_{\alpha}^{A'}$, $\bar{\zeta}_{A'}^{\dot{\alpha}}$ are decomposed in the same way. The parallel spinor

condition (4.18) for $\bar{\zeta}$, $\bar{\zeta}_{mn}$ and ζ_m are

$$\left[\delta^{\dot{\beta}}_{A'} \Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}} + \delta^{\dot{\alpha}}_{B'} K_{a,\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}{}^{B'} \right] \bar{\zeta} = 0, \quad (4.37)$$

$$\left[\delta^{\dot{\gamma}}_{A'} \Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}} (\bar{\sigma}^{pq})^{\dot{\beta}}_{\dot{\gamma}} + \delta^{\dot{\beta}}_{B'} K_{a,\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}{}^{B'} (\bar{\sigma}^{pq})^{\dot{\alpha}}_{\dot{\beta}} \right] \bar{\zeta}_{pq} = 0, \quad (4.38)$$

$$\left[\delta^{\dot{\gamma}}_{C'} \Omega_{mna} (\sigma^{mn})_{\alpha}{}^{\beta} (\sigma^p)_{\beta\dot{\gamma}} \epsilon^{A'C'} + \delta^{\dot{\beta}}_{C'} K_{a,\hat{b}\hat{c}} (\sigma^p)_{\alpha\dot{\beta}} \epsilon^{B'C'} (\Sigma^{\hat{b}\hat{c}})^{A'}{}_{B'} \right] \zeta_p = 0. \quad (4.39)$$

In order that the scalar supersymmetry is preserved, the $U(1)_R$ and $U(1)_{R'}$ charges must be identified as

$$\delta^{\dot{\alpha}}_{B'} K_{a,\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}{}^{B'} = -\delta^{\dot{\beta}}_{A'} \Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}. \quad (4.40)$$

We then find that the scalar and one component of the tensor supersymmetries are preserved and the others are broken under the condition (4.40)¹.

We then solve the constraints on the torsion. The anti-self-dual part of $K_{a,\hat{b}\hat{c}}$ and Ω_{mna} are identified by the relation (4.40). Then the anti-self-dual part of $K_{a',\hat{b}\hat{c}}$ is non-zero since $\epsilon_{1a'}$ and $\epsilon_{2a'}$ are non-zero. Therefore we find that $\Omega_{mn\hat{c}} = 0$ from (4.29) and (4.30) is satisfied automatically for any $K_{\hat{a},\hat{b}\hat{c}}$. Using (4.40), the anti-self-dual part of $K_{\hat{a},\hat{b}\hat{c}}$ becomes zero. Since the $U(1)_{L'} \times U(1)_{R'}$ $K_{a',\hat{b}\hat{c}}$ charge of $T_{\hat{a}\hat{b}}^{\hat{c}}$ is non-zero, the condition (4.35) implies $T_{\hat{a}\hat{b}}^{\hat{c}} = 0$. Then, using the relation (2.7) the self-dual part of $K_{\hat{a},\hat{b}\hat{c}}$ is zero. The conditions (4.33) and (4.36) are satisfied automatically. The self-dual part of $K_{a',\hat{b}\hat{c}}$ belongs to $U(1)_{L'}$. In summary, we have the following conditions on the Ω -background parameters Ω_{mna} and the contorsion $K_{a,bc}$ for the scalar supersymmetry generated by \bar{Q} :

$$\begin{aligned} \delta^{\dot{\alpha}}_{B'} K_{a',\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}{}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\ K_{a',\hat{b}\hat{c}} (\Sigma^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} &= -4i M_{a'}^{\hat{A}}_{\hat{B}}, \quad M_{a'}^{\hat{A}}_{\hat{B}} = \begin{pmatrix} m_{a'} & 0 \\ 0 & -m_{a'} \end{pmatrix}, \\ \Omega_{mn\hat{a}} &= K_{\hat{a},\hat{b}\hat{c}} = K_{a,b'c'} = K_{a,\hat{b}c'} = 0, \end{aligned} \quad (4.41)$$

where $m_{a'}$ are real parameters. The theory has $\mathcal{N} = (0, 2)$ supersymmetry² under the conditions (4.41). The explicit form of the supersymmetry transformation of fields are obtained by substituting (4.41) into (3.10). The result coincides with the transformation obtained in [12].

¹We will discuss the tensor supersymmetries in section 4.4

²The notation $\mathcal{N} = (m, n)$ means that the theory has m chiral, n anti-chiral supercharges.

The R-symmetry Wilson line $(\mathcal{A}_{a'})^{\hat{A}}_{\hat{B}}$ and the contorsion $K_{a',\hat{b}\hat{c}}$ are related by (4.6). The conditions on the contorsion in (4.41) are rewritten as

$$\begin{aligned} 4i\delta^{\dot{\alpha}}_{B'}(\mathcal{A})_{A'}^{B'} &= \delta^{\dot{\beta}}_{A'}\Omega_{mn}(\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, & 4i\delta^{\dot{\alpha}}_{B'}(\bar{\mathcal{A}})_{A'}^{B'} &= \delta^{\dot{\beta}}_{A'}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\ (\mathcal{A})^{\hat{A}}_{\hat{B}} &= \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, & (\bar{\mathcal{A}})^{\hat{A}}_{\hat{B}} &= \begin{pmatrix} \bar{m} & 0 \\ 0 & -\bar{m} \end{pmatrix}, \\ \Omega_{mn\hat{a}} &= (\mathcal{A}_{\hat{a}})^A_B(\Sigma_{\hat{b}\hat{c}})^B_A = (\mathcal{A}_a)^A_B(\Sigma_{b'c'})^B_A = (\mathcal{A}_a)^A_B(\Sigma_{\hat{b}\hat{c}})^B_A = 0, \end{aligned} \quad (4.42)$$

where $\mathcal{A} = \frac{1}{\sqrt{2}}(\mathcal{A}_5 - i\mathcal{A}_6)$, $\bar{\mathcal{A}} = \frac{1}{\sqrt{2}}(\mathcal{A}_5 + i\mathcal{A}_6)$ and $m, \bar{m}, \Omega_{mn}, \bar{\Omega}_{mn}$ are defined similarly. $(\mathcal{A})^{\hat{A}}_{\hat{B}}, (\bar{\mathcal{A}})^{\hat{A}}_{\hat{B}}$ are identified with the mass matrices of the hypermultiplet in the $\mathcal{N} = 2^*$ theory [4, 12]. The mass of the hypermultiplet is $\sqrt{m\bar{m}}$. Mass perturbations in twisted $\mathcal{N} = 4$ theory are discussed in [23].

4.3.2 Vafa-Witten twist

We next consider the Vafa-Witten twist. The supercharges $Q_{\alpha}^{A'}, \bar{Q}_{A'}^{\dot{\alpha}}$ and the parameters $\zeta_{\alpha}^{A'}, \bar{\zeta}_{A'}^{\dot{\alpha}}$ are decomposed as in the case of the half twist. The supercharges $Q_{\alpha}^{\hat{A}}, \bar{Q}_{\hat{A}}^{\dot{\alpha}}$ are decomposed into \bar{Q}', \bar{Q}'_{mn} and \bar{Q}'_{mn} . The parameters $\zeta_{\alpha}^{\hat{A}}, \bar{\zeta}_{\hat{A}}^{\dot{\alpha}}$ are decomposed in the same way. The parallel spinor conditions for $\bar{\zeta}, \bar{\zeta}_{mn}$ and ζ_m are (4.37)–(4.39). The conditions for $\bar{\zeta}', \bar{\zeta}'_{mn}$ and ζ'_m are

$$\left[\delta^{\dot{\beta}}_{\hat{A}}\Omega_{mna}(\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}} + \delta^{\dot{\alpha}}_{\hat{B}}K_{a,\hat{b}\hat{c}}(\bar{\Sigma}^{\hat{b}\hat{c}})_{\hat{A}}^{\hat{B}} \right] \bar{\zeta}' = 0, \quad (4.43)$$

$$\left[\delta^{\dot{\gamma}}_{\hat{A}}\Omega_{mna}(\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}(\bar{\sigma}^{pq})^{\dot{\beta}}_{\dot{\gamma}} + \delta^{\dot{\beta}}_{\hat{B}}K_{a,\hat{b}\hat{c}}(\bar{\Sigma}^{\hat{b}\hat{c}})_{\hat{A}}^{\hat{B}}(\bar{\sigma}^{pq})^{\dot{\alpha}}_{\dot{\beta}} \right] \bar{\zeta}'_{pq} = 0, \quad (4.44)$$

$$\left[\delta^{\dot{\gamma}}_{\hat{C}}\Omega_{mna}(\sigma^{mn})_{\alpha}^{\beta}(\sigma^p)_{\beta\dot{\gamma}}\epsilon^{\hat{A}\hat{C}} + \delta^{\dot{\beta}}_{\hat{C}}K_{a,\hat{b}\hat{c}}(\sigma^p)_{\alpha\dot{\beta}}\epsilon^{\hat{B}\hat{C}}(\Sigma^{\hat{b}\hat{c}})_{\hat{B}}^{\hat{A}} \right] \zeta'_p = 0. \quad (4.45)$$

The condition on the torsion for the scalar supersymmetry $\bar{\zeta}'$ is

$$\delta^{\dot{\alpha}}_{\hat{B}}K_{a,\hat{b}\hat{c}}(\bar{\Sigma}^{\hat{b}\hat{c}})_{\hat{A}}^{\hat{B}} = -\delta^{\dot{\beta}}_{\hat{A}}\Omega_{mna}(\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}. \quad (4.46)$$

The condition (4.46) together with (4.40) implies that the rank of the contorsion matrices $K_{a',\hat{b}\hat{c}}$ reduces by two and $K_{a',\hat{b}\hat{c}}$ can be of the form

$$K_{a',\hat{b}\hat{c}} = \begin{pmatrix} 0 & -k_{a'} & 0 & 0 \\ k_{a'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.47)$$

where $k_{a'}$ are non-zero parameters. Therefore, two matrices within $\Omega_{mn\hat{a}}$ ($\hat{a} = 7, 8, 9, 10$) remain non-zero from the condition (4.29). From the representation of the Σ -matrices in the appendix, we can take Ω_{mn9} , Ω_{mn10} as non-zero matrices and Ω_{mn7} and Ω_{mn8} as zero. Then from (4.40) and (4.46) we get $K_{\hat{a},\hat{b}\hat{c}} = 0$ except $K_{\hat{a},78}$ ($\hat{a} = 9, 10$) and the condition (4.30) is satisfied automatically. The condition (4.34) holds since the contorsion and the Ω -background matrices are identified by the relations (4.46) and (4.40), and the matrices Ω_{mna} commute with each other. We find that the condition (4.35) is satisfied and the (4.33) reduces to the commutative relation of Ω_{mn7} and Ω_{mn8} . The last condition (4.36) holds when the conditions (4.29), (4.30), (4.34) and (4.35) are satisfied.

We obtain the following conditions on the Ω -background parameters Ω_{mna} and the contorsion $K_{a,bc}$ for the scalar supersymmetries generated by \bar{Q}, \bar{Q}' :

$$\begin{aligned}
\delta^{\dot{\alpha}}_{B'} K_{a',\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\
\delta^{\dot{\alpha}}_{\hat{B}} K_{a',\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{\hat{A}}^{\hat{B}} &= -\delta^{\dot{\beta}}_{\hat{A}} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \quad (a' = 5, 6) \\
\delta^{\dot{\alpha}}_{B'} K_{\hat{a},\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mn\hat{a}} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\
\delta^{\dot{\alpha}}_{\hat{B}} K_{\hat{a},\hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{\hat{A}}^{\hat{B}} &= -\delta^{\dot{\beta}}_{\hat{A}} \Omega_{mn\hat{a}} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \quad (\hat{a} = 9, 10) \\
\Omega_{mn7} &= \Omega_{mn8} = K_{7,\hat{b}\hat{c}} = K_{8,\hat{b}\hat{c}} = K_{a,b'c'} = K_{a,\hat{b}c'} = 0.
\end{aligned} \tag{4.48}$$

In terms of the R-symmetry Wilson line, these become

$$\begin{aligned}
4i\delta^{\dot{\alpha}}_{B'} (\mathcal{A}_{a'})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\
4i\delta^{\dot{\alpha}}_{\hat{B}} (\mathcal{A}_{\hat{a}})_{\hat{A}}^{\hat{B}} &= -\delta^{\dot{\beta}}_{\hat{A}} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \quad (a' = 5, 6) \\
4i\delta^{\dot{\alpha}}_{B'} (\mathcal{A}_{a'})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mn\hat{a}} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\
4i\delta^{\dot{\alpha}}_{\hat{B}} (\mathcal{A}_{\hat{a}})_{\hat{A}}^{\hat{B}} &= -\delta^{\dot{\beta}}_{\hat{A}} \Omega_{mn\hat{a}} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \quad (\hat{a} = 9, 10), \\
(\mathcal{A}_7)^A_B (\Sigma_{\hat{b}\hat{c}})^B_A &= (\mathcal{A}_8)^A_B (\Sigma_{\hat{b}\hat{c}})^B_A = (\mathcal{A}_a)^A_B (\Sigma_{b'c'})^B_A = (\mathcal{A}_a)^A_B (\Sigma_{\hat{b}c'})^B_A = 0, \\
\Omega_{mn7} &= \Omega_{mn8} = 0.
\end{aligned} \tag{4.49}$$

As in the case of the half twist, two components of the tensor supersymmetries are preserved when the conditions (4.48), (4.49) are satisfied. Therefore the theory has $\mathcal{N} = (0, 4)$ supersymmetry.

4.3.3 Marcus twist

Finally, we consider the Marcus twist. The supercharges $Q_\alpha^{A'}, \bar{Q}^{\dot{A}'}_{A'}$ and $Q_\alpha^{\hat{A}}, \bar{Q}^{\dot{\hat{A}}}_{\hat{A}}$ are decomposed into $Q, \bar{Q}, Q_{mn}, \bar{Q}_{mn}$ and Q_m, \bar{Q}_m . The parameters $\zeta_\alpha^{A'}, \bar{\zeta}_{A'}^{\dot{A}'}$ and $\zeta_\alpha^{\hat{A}}, \bar{\zeta}_{\hat{A}}^{\dot{\hat{A}}}$ are decomposed similarly. The parallel spinor conditions for $\bar{\zeta}, \bar{\zeta}_{mn}$ and ζ_m are (4.37)–(4.39). The condition (4.18) for ζ, ζ_{pq} and $\bar{\zeta}_p$ are

$$\left[\delta_\beta^{\hat{A}} \Omega_{mna} (\sigma^{mn})_\alpha^\beta + \delta_\alpha^{\hat{B}} K_{a, \hat{b}\hat{c}} (\Sigma^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} \right] \zeta = 0, \quad (4.50)$$

$$\left[\delta_\alpha^{\hat{A}} \Omega_{mna} (\sigma^{mn})_\alpha^\beta (\sigma^{pq})_\beta^\alpha + \delta_\beta^{\hat{B}} K_{a, \hat{b}\hat{c}} (\sigma^{pq})_\alpha^\beta (\Sigma^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} \right] \zeta_{pq} = 0, \quad (4.51)$$

$$\left[\delta_\gamma^{\hat{C}} \Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}} (\sigma^p)^{\dot{\beta}\gamma} \epsilon_{\hat{A}\hat{C}} + \delta_\beta^{\hat{C}} K_{a, \hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} (\sigma^p)^{\dot{\alpha}\beta} \epsilon_{\hat{B}\hat{C}} \right] \bar{\zeta}_p = 0. \quad (4.52)$$

We examine the conditions on the two scalar supersymmetries generated by Q, \bar{Q} . The condition on the torsion for the parallel spinor ζ is

$$\delta_\alpha^{\hat{B}} K_{a, \hat{b}\hat{c}} (\Sigma^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} = -\delta_\beta^{\hat{A}} \Omega_{mna} (\sigma^{mn})_\alpha^\beta. \quad (4.53)$$

Using the relation (4.53) together with (4.40), the condition (4.29) implies that the matrices $\Omega_{mn\hat{c}}$ vanish. Then the condition (4.30) holds automatically. The condition (4.34) is satisfied by using the relation (4.53). Similarly $K_{\hat{a}, \hat{b}\hat{c}}$ is shown to be zero from $\Omega_{mn\hat{c}} = 0$. Then the conditions (4.35), (4.34) and (4.36) are satisfied.

We get the following conditions on the Ω -background parameters Ω_{mna} and the torsion $K_{a, bc}$ for the scalar supersymmetries generated by Q, \bar{Q} :

$$\begin{aligned} \delta^{\dot{\alpha}}_{B'} K_{a', \hat{b}\hat{c}} (\bar{\Sigma}^{\hat{b}\hat{c}})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\ \delta_\alpha^{\hat{B}} K_{a', \hat{b}\hat{c}} (\Sigma^{\hat{b}\hat{c}})^{\hat{A}}_{\hat{B}} &= -\delta_\beta^{\hat{A}} \Omega_{mna'} (\sigma^{mn})_\alpha^\beta, \\ \Omega_{mn\hat{a}} &= K_{\hat{a}, \hat{b}\hat{c}} = K_{a, b'c'} = K_{a, \hat{b}c'} = 0. \end{aligned} \quad (4.54)$$

In terms of the R-symmetry Wilson line, the conditions become

$$\begin{aligned} 4i\delta^{\dot{\alpha}}_{B'} (\mathcal{A}_{a'})_{A'}^{B'} &= -\delta^{\dot{\beta}}_{A'} \Omega_{mna'} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}}, \\ 4i\delta_\alpha^{\hat{B}} (\mathcal{A}_{a'})^{\hat{A}}_{\hat{B}} &= -\delta_\beta^{\hat{A}} \Omega_{mna'} (\sigma^{mn})_\alpha^\beta, \\ \Omega_{mn\hat{a}} &= (\mathcal{A}_{\hat{a}})^A_B (\Sigma_{\hat{b}\hat{c}})^B_A = (\mathcal{A}_a)^A_B (\Sigma_{b'c'})^B_A = (\mathcal{A}_a)^A_B (\Sigma_{\hat{b}c'})^B_A = 0. \end{aligned} \quad (4.55)$$

In addition to the scalar supersymmetries, two components of the tensor supersymmetries are preserved when the conditions (4.54), (4.55) are satisfied. Therefore the theory has $\mathcal{N} = (2, 2)$ supersymmetry.

4.4 Nekrasov-Shatashvili limit

We have examined the scalar supersymmetries of the $\mathcal{N} = 4$ super Yang-Mills theory in the Ω -background with the torsion. In this subsection, we study supersymmetries of the theory in the Nekrasov-Shatashvili limit of the Ω -background [18]. It is defined by the limit where ϵ_{2a} (or ϵ_{1a}) $\rightarrow 0$ and keeping ϵ_{1a} (or ϵ_{2a}) finite. In this limit, the super Poincaré symmetry of the two-dimensional subspace in four-dimensional spacetime is recovered. We will study how supersymmetry is enhanced in each topological twist.

Half twist First, we consider the half twist where the contorsion and the Ω -background matrices are related by (4.40). We examine the parallel spinor conditions for the tensor and the vector supersymmetries (4.38), (4.39).

For the vector supersymmetry, by eliminating the contorsion in (4.39), we get

$$\left[\Omega_{mna} (\sigma^{mn})_{\alpha}{}^{\beta} (\sigma^p)_{\beta\dot{\beta}} \varepsilon^{\dot{\beta}A'} + (\sigma^p)_{\alpha\dot{\alpha}} \varepsilon^{\dot{\alpha}B'} \Omega_{mna} (\bar{\sigma}^{mn})_{B'}{}^{A'} \right] \zeta_p = 0. \quad (4.56)$$

This equation (4.56) is written in terms of $\epsilon_{1a}, \epsilon_{2a}$ as

$$\epsilon_{1a} (\sigma^4 \zeta_1 - \sigma^3 \zeta_2) + \epsilon_{2a} (\sigma^2 \zeta_3 - \sigma^1 \zeta_4) = 0. \quad (4.57)$$

In the Nekrasov-Shatashvili limit, $\epsilon_{2a} \rightarrow 0$ ($a = 5, 6$), the parameters ζ_3, ζ_4 satisfy the parallel spinor condition. In the limit $\epsilon_{1a} \rightarrow 0$ ($a = 5, 6$), the parameters ζ_1, ζ_2 satisfy the condition.

We next consider parallel spinor condition for $\bar{\zeta}_{pq}$. Using the relation (4.40), the condition is rewritten as

$$\left[\Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} (\bar{\sigma}^{pq})^{\dot{\beta}}{}_{A'} - (\bar{\sigma}^{pq})^{\dot{\alpha}}{}_{\dot{\beta}} \Omega_{mna} (\bar{\sigma}^{mn})^{\dot{\beta}}{}_{A'} \right] \bar{\zeta}_{pq} = 0. \quad (4.58)$$

This is the commutative relation between the matrices $\Omega_{mna}(\bar{\sigma}^{mn})$ and $\bar{\zeta}_{pq}(\bar{\sigma}^{pq})$,

$$[\Omega_{mna} \bar{\sigma}^{mn}, \bar{\zeta}_{pq} \bar{\sigma}^{pq}] = 0. \quad (4.59)$$

Since we have $\Omega_{mna}(\bar{\sigma}^{mn}) = i(\epsilon_{1a} + \epsilon_{2a})\tau_3$, the parameter $\bar{\zeta}_{12}$ satisfies (4.59) as mentioned in section 4.3.1 and $\bar{\zeta}_{13} = \bar{\zeta}_{14} = 0$. Therefore $\mathcal{N} = (2, 2)$ supersymmetry is preserved in the Nekrasov-Shatashvili limit. The conserved supercharges of the $\mathcal{N} = (2, 2)$ supersymmetry in the half twist case is summarized in table 3.

Supercharge	Scalar	Tensor	Vector
$\epsilon_{1a} \rightarrow 0 \ (a = 5, 6)$	\bar{Q}	\bar{Q}_{12}	Q_1, Q_2
$\epsilon_{2a} \rightarrow 0 \ (a = 5, 6)$	\bar{Q}	\bar{Q}_{12}	Q_3, Q_4

Table 3: Conserved supercharges in the half twist.

Supercharge	Scalar	Tensor	Vector
$\epsilon_{1a} \rightarrow 0 \ (a = 5, 6, 7, 8)$	\bar{Q}, \bar{Q}'	$\bar{Q}_{12}, \bar{Q}'_{12}$	Q_1, Q_2, Q'_1, Q'_2
$\epsilon_{2a} \rightarrow 0 \ (a = 5, 6, 7, 8)$	\bar{Q}, \bar{Q}'	$\bar{Q}_{12}, \bar{Q}'_{12}$	Q_3, Q_4, Q'_3, Q'_4

Table 4: Conserved supercharges in the Vafa-Witten twist.

Vafa-Witten twist . The parallel spinor conditions for $\bar{\zeta}'_{mn}$ and ζ'_m are the same as those for $\bar{\zeta}_{mn}$ and ζ_m in the half twist. Therefore $\mathcal{N} = (4, 4)$ supersymmetry is preserved in the Nekrasov-Shatashvili limit. The conserved supercharges are found in table 4.

Marcus twist The parallel spinor conditions for $\bar{\zeta}_{pq}$, ζ_p have been written down. The parallel spinor condition for ζ_{pq} is obtained by replacing $\bar{\sigma}_{mn}$ with σ_{mn} in (4.59). Then ζ_{12} satisfies the equation (4.59) and $\zeta_{13} = \zeta_{14} = 0$. The condition for $\bar{\zeta}_p$ is

$$\epsilon_{1a}(\sigma^2 \bar{\zeta}_3 - \sigma^1 \bar{\zeta}_4) + \epsilon_{2a}(\sigma^4 \bar{\zeta}_1 - \sigma^3 \bar{\zeta}_2) = 0. \quad (4.60)$$

Therefore $\mathcal{N} = (4, 4)$ supersymmetry is preserved in the Nekrasov-Shatashvili limit. The conserved supercharges are summarized in table 5.

5 Conclusions and Discussion

In this paper we studied ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in the curved background with the torsion. We investigated the dimensional reduction to four dimensions where the torsion is introduced along the internal space such that four-dimensional gauge invariance is preserved. Requiring supersymmetry, it has been shown that the torsion obeys the constraints and also modifies the parallel spinor conditions. In particular we have studied the torsion in the ten-dimensional Ω -background, where we can identify the R-symmetry Wilson line gauge field with the torsion. We solved the modified parallel

Supercharge	Scalar	Tensor	Vector
$\epsilon_{1a} \rightarrow 0 \ (a = 5, 6)$	\bar{Q}, Q	\bar{Q}_{12}, Q_{12}	$Q_1, Q_2, \bar{Q}_1, \bar{Q}_2$
$\epsilon_{2a} \rightarrow 0 \ (a = 5, 6)$	\bar{Q}, Q	\bar{Q}_{12}, Q_{12}	$Q_3, Q_4, \bar{Q}_3, \bar{Q}_4$

Table 5: Conserved supercharges in the Marcus twist.

spinor conditions for the topological twists (the half-twist, the Vafa-Witten twist and the Marcus twist) of $\mathcal{N} = 4$ supersymmetry. We found the solutions of the deformation parameters of the Ω -background and the Wilson line gauge fields. We obtained the preserved supersymmetries for these twists.

In order to construct the deformed topological field theory associated with the twists, it is necessary to extend the deformed scalar supersymmetry to the off-shell supersymmetry. In a subsequent paper [24], we will discuss the off-shell structure of the twisted deformed theories and their instanton effective action.

One can consider the curved background admitting the (conformal) Killing spinor associated with super(conformal)symmetry transformations. In the case without the torsion, the parallel spinor conditions lead to the Ricci-flatness of the geometry. For the case with the Killing spinor, the associated geometries have been classified in [14]. The conformal Killing spinor in S^4 [15, 16] and other geometries [25, 26, 27] are used to discuss the localization. It would be interesting to introduce the torsion for the geometry admitting (conformal) Killing spinor conditions and examine the deformed super(conformal)symmetry.

It would also be an interesting problem to understand the ten-dimensional deformed theory in superstring theory. The stringy realization of the Ω -background allows us to analyze the various dimensional system [28, 29, 30, 31, 32, 33]. From the dimensional reduction to various dimensions, we can obtain the Ω -deformed gauge theories other than four dimensions in a systematic way. One can also study their Nekrasov-Shatashvili limit, where the various Ω -deformed BPS states exist in the deformed theories [34, 35, 36, 37]. From the supersymmetry constructed in this work, one can compute the central charges for the BPS states, which are important to understand the integrability structure of the deformed theory.

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Appendix Dirac matrices in four and six dimensions

In this appendix, we present our conventions of the Dirac matrices in four- and six-dimensional spaces with the Euclidean signature. The Dirac matrices $\sigma_{\alpha\dot{\alpha}}^m$ and $\bar{\sigma}^{m\dot{\alpha}\alpha}$ in four dimensions are defined by

$$\sigma^m = (i\tau^1, i\tau^2, i\tau^3, \mathbf{1}_2), \quad \bar{\sigma}^m = (-i\tau^1, -i\tau^2, -i\tau^3, \mathbf{1}_2), \quad (\text{A.1})$$

where τ^i ($i = 1, 2, 3$) are the Pauli matrices and $\mathbf{1}_2$ denotes the 2×2 identity matrix. We define the Lorentz generators σ^{mn} and $\bar{\sigma}^{mn}$ by

$$\sigma^{mn} = \frac{1}{4}(\sigma^m \bar{\sigma}^n - \sigma^n \bar{\sigma}^m), \quad \bar{\sigma}^{mn} = \frac{1}{4}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m). \quad (\text{A.2})$$

The Dirac matrices Σ^{aAB} and $\bar{\Sigma}_{AB}^a$ in six dimensions are defined by

$$\begin{aligned} \Sigma^5 &= \begin{pmatrix} i\tau^2 & 0 \\ 0 & i\tau^2 \end{pmatrix}, & \Sigma^6 &= \begin{pmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix}, & \Sigma^7 &= \begin{pmatrix} 0 & -\tau^3 \\ \tau^3 & 0 \end{pmatrix}, \\ \Sigma^8 &= \begin{pmatrix} 0 & i\mathbf{1}_2 \\ -i\mathbf{1}_2 & 0 \end{pmatrix}, & \Sigma^9 &= \begin{pmatrix} 0 & -\tau^1 \\ \tau^1 & 0 \end{pmatrix}, & \Sigma^{10} &= \begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 0 \end{pmatrix}, \\ \bar{\Sigma}^5 &= \begin{pmatrix} -i\tau^2 & 0 \\ 0 & -i\tau^2 \end{pmatrix}, & \bar{\Sigma}^6 &= \begin{pmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix}, & \bar{\Sigma}^7 &= \begin{pmatrix} 0 & \tau^3 \\ -\tau^3 & 0 \end{pmatrix}, \\ \bar{\Sigma}^8 &= \begin{pmatrix} 0 & i\mathbf{1}_2 \\ -i\mathbf{1}_2 & 0 \end{pmatrix}, & \bar{\Sigma}^9 &= \begin{pmatrix} 0 & \tau^1 \\ -\tau^1 & 0 \end{pmatrix}, & \bar{\Sigma}^{10} &= \begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A.3})$$

The Lorentz generators Σ^{ab} and $\bar{\Sigma}^{ab}$ are defined by

$$\Sigma^{ab} = \frac{1}{4}(\Sigma^a \bar{\Sigma}^b - \Sigma^b \bar{\Sigma}^a), \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\bar{\Sigma}^a \Sigma^b - \bar{\Sigma}^b \Sigma^a). \quad (\text{A.4})$$

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